ON THE THEORY OF RANDOMIZED RESPONSES TO TWO SENSITIVE QUESTIONS*

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This paper sets forth a stochastic model which can be applied to randomized response designs wherein more than one question is sensitive to stigmatizing. Using three specific designs applied to a pair of stigmatizing or sensitive traits leading to two by two contingency tables, it is proved that if the two traits are statistically independent then the responses are also and conversely. The converse statement gives the support we need for inference to be made on the traits based on the randomized responses.

INTRODUCTION

Let Π be a vector of *trait* probabilities which are unknown; let Λ be a corresponding vector of *response* probabilities; and let P be a nonsingular *randomizing* matrix. By definition we set $\Lambda = P\Pi$. Since Λ is estimated directly from the responses, $\hat{\Pi} = P^{-1}\hat{\Lambda}$ and $\text{Cov}(\hat{\Pi}) = P^{-1}\Sigma P^{-1'}$ wherein $\Sigma = \text{Cov}(\hat{\Lambda})$.

This differs from the Warner (1971) approach in that his P matrix is the expectation of a stochastic matrix, and P is singular "requiring" weighted least squares to estimate Π. In the present model P is a transition matrix between two state vectors represented by Π and Λ, respectively, the trait and response state probabilities. An element of P, p_{ij}, is the probability of obtaining the response i, given a subject is in state j.

Finally, Π and Λ are vectors of cell probabilities of contingency tables. Using the initial Warner design (1965), the Simmons or Unrelated Question design (Greenberg, et al. 1969) and the Forced Yes design (Drane, 1975) it is proved that "if the two (stigmatizing) traits are independent, then the responses are also and conversely."

It is the converse that is essential to proper inferences being drawn from analyses of contingency tables constructed as the result of a randomized response design survey.

1. A STOCHASTIC MODEL FOR TWO SENSITIVE QUESTIONS

Let us consider two sensitive or stigmatizing traits and two randomized questions, each directed at only one of these traits. For the purposes of this treatment the questions and their respective randomizing devices will give rise to YES-NO answers, only. Presuming independence between trials and a random selection of individuals, each respondent produces an entry to a 2×2 contingency table. If π_j are the trait probabilities, λ_i the response probabilities and p_{ij} the conditional probability that a person in state j will produce a response i, then the randomized response can be summarized as

$$\Lambda = P\Pi \tag{1.1}$$

or, by resubscripting according to 2×2 tables,

$$\begin{bmatrix}\lambda_{11}\\ \lambda_{12}\\ \lambda_{21}\\ \lambda_{22}\end{bmatrix} = P\begin{bmatrix}\pi_{11}\\ \pi_{12}\\ \pi_{21}\\ \pi_{22}\end{bmatrix} \tag{1.2}$$

wherein P is a 4×4 randomizing or transition matrix. Furthermore we shall require P to be nonsingular. The 2×2 Trait Table will be that of Figure 1 with one for one substitutions when considering the corresponding 2×2 Table.

		Trait 2: Y	N	
Trait 1	Y	π_{11}	π_{12}	$\pi_{1.}$
	N	π_{21}	π_{22}	$\pi_{2.}$
		$\pi_{.1}$	$\pi_{.2}$	1

FIG. 1

Table of Trait Probabilities

In the figure, π_{11} = probability a person drawn at random possesses both traits, π_{12} = probability that he possesses the first but not the second, etc. The $\pi_{i.}$ and $\pi_{.j}$ are the corresponding marginal probabilities. The table of response probabilities is obtained by substituting λ everywhere for π and "Response" for "Trait."

The requirement that P be nonsingular results from the fact that inferences on Π are made from observations made with probabilities Λ. Thus, if information about Λ, say $\hat{\Lambda}$, is to be preserved with no loss whatever, then not only should (1.1) and (1.2) hold, but

$$\Pi = P^{-1}\Lambda \tag{1.3}$$

should also be true, so that

$$\hat{\Pi} = P^{-1}\hat{\Lambda} \tag{1.4}$$

which is a restatement of the invariance principle of maximum likelihood estimates.

The usual hypothesis for independence is that Π can be written as a vector of products, namely

$$\Pi = \begin{pmatrix}\pi_{11}\\ \pi_{12}\\ \pi_{21}\\ \pi_{22}\end{pmatrix} = \begin{pmatrix}\pi_{1.} & \pi_{.1}\\ \pi_{1.} & \pi_{.2}\\ \pi_{2.} & \pi_{.1}\\ \pi_{2.} & \pi_{.2}\end{pmatrix} \tag{1.5}$$

If the two stigmatizing traits are independent, then (1.5) will be true. Once again, however, we are not observing responses to direct questions. So the appropriate question is the following: If the responses are independent, are the traits also

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independent? The following theorem will be proved for three different designs previously proposed and/or used.

Theorem: If the two (stigmatizing) traits are independent, then the responses are also and conversely.

2. THREE DESIGNS

2.1. Forced Yes Design

The first design is a simplified version of the Morton design (see Greenberg, Horvitz and Abernathy, 1974, and Drane, 1975) in which the alternative question always has the same positive answer "Yes" or the person being questioned is required to answer "Yes".

Definitions:

p_i = probability of being asked the ith sensitive question, $i = 1, 2$,

q_i = probability of being instructed to answer "Yes" on the ith question and

π_{ij}, $\pi_{.j}$ and $\pi_{i.}$ are the trait probabilities of the 2×2 table of Figure 1.

The response table, with the same form as that of Figure 1, has entries λ_{ij} as follows:

$$\lambda_{11} = p_1 p_2 \pi_{11} + p_1 q_2 \pi_{1.} + q_1 p_2 \pi_{.1} + q_1 q_2 \pi_{..}\ , \tag{2.1}$$

where $p_1 p_2 \pi_{11}$ is the probability that the subject will be asked both stigmatizing questions and that the answer to both will be "Yes;" $p_1 q_2 \pi_{1.}$ is the probability that only the first stigmatizing question will be asked and the response will be "Yes", forcing a "yes" instead of the second question; $q_1 p_2 \pi_{.1}$ is the reverse; and $q_1 q_2 \pi_{..}$ is the probability that each answer will be a forced yes and $\pi_{..} = \pi_{11} + \pi_{12} + \pi_{21} + \pi_{22} = 1$. Substituting for $\pi_{1.}$, $\pi_{.1}$ and $\pi_{..}$ we get,

$$\lambda_{11} = \pi_{11} + q_2 \pi_{12} + q_1 \pi_{21} + q_1 q_2 \pi_{22}. \tag{2.2}$$

In a manner similar to that used in deriving (2.1) and (2.2) we obtain the matrix equation (1.1) wherein

$$P = \begin{pmatrix} 1 & q_2 & q_1 & q_1 q_2 \\ 0 & p_2 & 0 & q_1 p_2 \\ 0 & 0 & p_1 & p_1 q_2 \\ 0 & 0 & 0 & p_1 p_2 \end{pmatrix}, \tag{2.3}$$

and

$$P^{-1} = \frac{1}{p_1 p_2} \begin{pmatrix} p_1 p_2 & -p_1 q_2 & -q_1 p_2 & q_1 q_2 \\ 0 & p_1 & 0 & -q_1 \\ 0 & 0 & p_2 & -q_2 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{2.4}$$

If we apply the hypothesis that $\pi_{ij} = \pi_{i.}\pi_{.j}$ and substitute this and (2.3) into (1.1), we obtain

$$\Lambda = \begin{pmatrix} (\pi_{1.} + q_1 \pi_{2.})(\pi_{.1} + q_2 \pi_{.2}) \\ (\pi_{1.} + q_1 \pi_{2.}) p_2 \pi_{.2} \\ p_1 \pi_{2.}(\pi_{.1} + q_2 \pi_{.2}) \\ p_1 \pi_{2.} p_2 \pi_{.2} \end{pmatrix}, \tag{2.5}$$

which proves the direct theorem. The converse theorem is proved by substituting (2.4) into (1.3) and assuming, $\lambda_{ij} = \lambda_{i.}\lambda_{.j}$, independence among the responses. In this case we obtain

$$\Pi = \frac{1}{p_1 p_2} \begin{pmatrix} (p_1 \lambda_{1.} - q_1 \lambda_{2.})(p_2 \lambda_{.1} - q_2 \lambda_{.2}) \\ (p_1 \lambda_{1.} - q_1 \lambda_{2.}) \lambda_{.2} \\ \lambda_{2.}(p_2 \lambda_{.1} - q_2 \lambda_{.2}) \\ \lambda_{2.} \lambda_{.2} \end{pmatrix} \tag{2.6}$$

which proves the converse theorem.

2.2. Initial Warner Design

Instead of the alternative being a forced Yes, Warner (1965) proposed an alternate question or statement which was the negative of the sensitive question. The respondent was asked to verify (Yes, No) one of the following: "I belong to Group A," or "I do not belong to Group A," with probabilities p_1 and q_1 respectively. We extend that design here to two questions and ask for verifications of "I belong to Group B" or "I do not belong to Group B" with probabilities p_2 and q_2. The trait and response probabilities, π_{ij} and λ_{ij}, are as before stated. Applying the step by step procedure as in the preceding example, we obtain

$$P = \begin{pmatrix} p_1 p_2 & p_1 q_2 & q_1 p_2 & q_1 q_2 \\ p_1 q_2 & p_1 p_2 & q_1 q_2 & q_1 p_2 \\ q_1 p_2 & q_1 q_2 & p_1 p_2 & p_1 q_2 \\ q_1 q_2 & q_1 p_2 & p_1 q_2 & p_1 p_2 \end{pmatrix}, \tag{2.7}$$

and

$$P^{-1} = \frac{1}{(p_1 - q_1)(p_2 - q_2)} \begin{pmatrix} p_1 p_2 & -p_1 q_2 & -q_1 p_2 & q_1 q_2 \\ -p_1 q_2 & p_1 p_2 & q_1 q_2 & -q_1 p_2 \\ -q_1 p_2 & q_1 q_2 & p_1 p_2 & -p_1 q_2 \\ q_1 q_2 & -q_1 p_2 & -p_1 q_2 & p_1 p_2 \end{pmatrix}. \tag{2.8}$$

Once again applying the hypothesis of independence to Π and Λ, (2.7) and (2.8) inserted into (1.1) and (1.3), respectively, yield

$$\Lambda = \begin{pmatrix} (p_1 \pi_{1.} + q_1 \pi_{2.})(p_2 \pi_{.1} + q_2 \pi_{.2}) \\ (p_1 \pi_{1.} + q_1 \pi_{2.})(q_2 \pi_{.1} + p_2 \pi_{.2}) \\ (q_1 \pi_{1.} + p_1 \pi_{2.})(p_2 \pi_{.1} + q_2 \pi_{.2}) \\ (q_1 \pi_{1.} + p_1 \pi_{2.})(q_2 \pi_{.1} + p_2 \pi_{.2}) \end{pmatrix} \tag{2.9}$$

and

$$\Pi = \frac{1}{(p_1-q_1)(p_2-q_2)}\begin{pmatrix} (p_1\lambda_{1.} - q_1\lambda_{2.})(p_2\lambda_{.1} - q_2\lambda_{.2}) \\ (p_1\lambda_{1.} - q_1\lambda_{2.})(-q_2\lambda_{.1} + p_2\lambda_{.2}) \\ (q_1\lambda_{1.} - p_1\lambda_{2.})(-p_2\lambda_{.1} + q_2\lambda_{.2}) \\ (q_1\lambda_{1.} - p_1\lambda_{2.})(q_2\lambda_{.1} - p_2\lambda_{.2}) \end{pmatrix} \tag{2.10}$$

which are proofs of both the theorem and its converse.

2.3. Unrelated Question Model

Walt Simmons suggested (Greenberg et al. 1969) that the second statement of Warner be replaced by an innocuous statement which was a priori unrelated to the stigmatizing statement, and which innocuous trait occurred with known probability. We again let

p_i = probability of being asked the ith sensitive question,
q_i = probability of being asked the ith innocuous question,
ϕ_i = probability of "Yes" to the ith innocuous question, and
$\theta_i = 1 - \phi_i$ (i = 1, 2 in all cases),

with Π and Λ as before. A given respondent may give up any combination of answers regardless of his true trait state. But the randomizing matrix is beginning to show some complexity.

$$P = \begin{pmatrix} (p_1+q_1\phi_1)(p_2+q_2\phi_2) & (p_1+q_1\phi_1)q_2\phi_2 & q_1\phi_1(p_2+q_2\phi_2) & q_1\phi_1q_2\phi_2 \\ (p_1+q_1\phi_1)q_2\phi_2 & (p_1+q_1\phi_1)(p_2+q_2\phi_2) & q_1\phi_1q_2\phi_2 & q_1\phi_1(p_2+q_2\phi_2) \\ q_1\phi_1(p_2+q_2\phi_2) & q_1\phi_1q_2\phi_2 & (p_1+q_1\phi_1)(p_2+q_2\phi_2) & (p_1+q_1\phi_1)q_2\phi_2 \\ q_1\phi_1q_2\phi_2 & q_1\phi_1(p_2+q_2\phi_2) & (p_1+q_1\phi_1)q_2\phi_2 & (p_1+q_1\phi_1)(p_2+q_2\phi_2) \end{pmatrix} \tag{2.11}$$

and

$$P^{-1} = 1/p_1p_2 \cdot \begin{pmatrix} (p_1+q_1\phi_1)(p_2+q_2\phi_2) & -(p_1+q_1\phi_1)q_2\phi_2 & -q_1\phi_1(p_2+q_2\phi_2) & q_1\phi_1q_2\phi_2 \\ -(p_1+q_1\phi_1)q_2\phi_2 & (p_1+q_1\phi_1)(p_2+q_2\phi_2) & q_1\phi_1q_2\phi_2 & -q_1\phi_1(p_2+q_2\phi_2) \\ -q_1\phi_1(p_2+q_2\phi_2) & q_1\phi_1q_2\phi_2 & (p_1+q_1\phi_1)(p_2+q_2\phi_2) & -(p_1+q_1\phi_1)q_2\phi_2 \\ q_1\phi_1q_2\phi_2 & -q_1\phi_1(p_2+q_2\phi_2) & -(p_1+q_1\phi_1)q_2\phi_2 & (p_1+q_1\phi_1)(p_2+q_2\phi_2) \end{pmatrix} \tag{2.12}$$

Finally, if $\pi_{ij} = \pi_{i.}\pi_{.j}$ and if $\lambda_{ij} = \lambda_{i.}\lambda_{.j}$, i, j = 1, 2, then

$$\Lambda = \begin{pmatrix} [(p_1+q_1\phi_1)\pi_{1.} + q_1\phi_1\pi_{2.}][(p_2+q_2\phi_2)\pi_{.1} + q_2\phi_2\pi_{.2}] \\ [(p_1+q_1\phi_1)\pi_{1.} + q_1\phi_1\pi_{2.}][q_2\theta_2\pi_{.1} + (p_2+q_2\theta_2)\pi_{.2}] \\ [q_1\theta_1\pi_{1.} + (p_1+q_1\theta_1)\pi_{2.}][(p_2+q_2\phi_2)\pi_{.1} + q_2\phi_2\pi_{.2}] \\ [q_1\theta_1\pi_{1.} + (p_1+q_1\theta_1)\pi_{2.}][q_2\theta_2\pi_{.1} + (p_2+q_2\theta_2)\pi_{.2}] \end{pmatrix} \tag{2.13}$$

and

$$\Pi = \frac{1}{p_1p_2}\begin{pmatrix} [(p_1+q_1\theta_1)\lambda_{1.} - q_1\phi_1\lambda_{2.}][(p_2+q_2\theta_2)\lambda_{.1} - q_2\phi_2\lambda_{.2}] \\ [-(p_1+q_1\theta_1)\lambda_{1.} + q_1\phi_1\lambda_{2.}][q_2\theta_2\lambda_{.1} - (p_2+q_2\phi_2)\lambda_{.2}] \\ [-q_1\theta_1\lambda_{1.} + (p_1+q_1\phi_1)\lambda_{2.}][(p_2+q_2\theta_2)\lambda_{.1} - q_2\phi_2\lambda_{.2}] \\ [q_1\theta_1\lambda_{1.} - (p_1+q_1\phi_1)\lambda_{2.}][q_2\theta_2\lambda_{.1} - (p_2+q_2\phi_2\lambda_{.2}] \end{pmatrix}, \tag{2.14}$$

which proves both the theorem and its converse for the Unrelated Question design.

3. SUMMARY

For all three designs independence between either traits or responses implies the other. The randomizing matrix P contains this information. The columns of P always add to one regardless of whether independence is preserved between Π and Λ. It is a requirement that the rows of P be the coefficients of a binomial product of the form

$$\lambda_{ij} = (a_i\pi_{1.} + b_i\pi_{2.})(c_i\pi_{.1} + d_i\pi_{.2}) \tag{3.1}$$

if independence between the traits imply independence between the responses. At this writing it is not known what is required of P so that Π also has row elements of the form of (3.1) with λ and π interchanged.

It is easy to see that the Unrelated Question design of Simmons reduces to the Forced Yes design by setting $\phi_i = 1$ and $\theta_i = 0$. The determinant of P in both cases is $(p_1p_2)^2$.

The requirement that $p_i \neq q_i$ in the Warner design is again in evidence. Neither of the other two designs reduces to this one by manipulating ϕ and θ.

Many other designs are left untouched by this paper, as is the connection between efficiency, power, etc. and P. These are left as objects of further investigations.

This paper has established that inferences regarding independence between the responses in a 2×2 table with either of the foregoing designs carries over to the traits as well.

Remarks:

Bill Parr, a student at Southern Methodist University, has pointed out that on purely logical grounds we have the following proposition: If independence among the traits implies independence among responses, then dependence among responses implies dependence among the traits. So that inferring dependence among responses carries over in this case, but it is possible without the above theorem to have independence among responses and dependence among traits.

Gould, Shah and Abernathy (1969) developed a behavioral model which fits into a 2×2 table and has 42 parameters in it which has to be some kind of a record. Then without asking questions of independence, as it was built on repeated sampling, they estimated as many as five parameters the rest being accounted for by constraints.

Lastly, bibliographies can be found in two expository articles by Greenberg, Horvitz and Abernathy (1974) and Horvitz, Greenberg and Abernathy (1975).

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